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## GARCH 模型在風險性債券評價之應用與信用價差期限結構分析

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# GARCH 模型在風險性債券評價之應用與信用價差期限結構分析

A GARCH Model for Pricing Risky Debt and the Term Structure of Credit Spreads


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## 摘要

本文發展一簡單 GARCH 模型來處理風險性債券的評價問題，文中同時討論利率風險和違約風險對債券價格的影響。藉由 GARCH 型評價模型，我們推導出可以刻畫浮動利率下的風險性債券的評價解析解(analytical solution)。文中並將對評價結果與在此模型下的信用風險避險效果進行分析。例如：本研究會討論利率水準高低與違約風險之間的相關性是否是決定信用價差大小的重要因素。我們利用既有之公司債券其收益率的市場資料參數值，對信用價差與利率水準二變數之相關性進行研究。同時，也探討風險性債券之存續期間與利率變動的關係。最後，檢視本研究所提之 GARCH 風險性債券模型的評價結果與實證資料所隱含的經濟意義是否一致。



關鍵字：GARCH 模型，違約風險，利率風險，信用價差。



## Abstract

This proposal tries to develop a GARCH model to valuing risky debt that accommodates both interest rate and credit risk. We use this methodology to infer analytical-form valuation expression for floating rate debt. This paper will provide a number of meaningful new insights about pricing and hedging credit risky instruments. For instance, we will discuss that the correlation between the interest rate and default risk whether has a significant effect on the properties of the credit spread. Using corporate bond yield data, paper will explore that credit spreads are negatively or non-negatively related to interest rates and that durations of risky bonds whether depend on the correlation with the changes of interest rates. Finally, explaining that the empirical evidence whether is consistent with the implications of the GARCH valuation model.



Keywords: GARCH Model, Default Risk, Interest Rate Risk, Credit Spread.

## 1. Introduction

Risk management has gone through a striking evolution from its origins in the 1970s to the current state-of-the-art systems and processes. After a decade of calm in the 1990s, following a turbulent decade in the 1980s, defaults of public companies in the U.S. began to surge. In addition, more than 100,000 unlisted companies went bankrupt in Japan during the last decade, also see Van Deventer and Imai(2003). It is not surprising that senior management at financial institutions around the world is highly focused on the most recent pronouncements of Basel Committee for Banking Supervision, particularly those that relate to credit risk.

Credit risk should be treated as part of market risk. The measurement of credit risk, however, provides its own set of challenges. Many credit-sensitive instruments are relatively illiquid, remain on a firm's books for lengthy periods of time, and cannot be reliably marked to market. The traditional Merton(1974) and Black and Cox(1976)contingent claims based approach to valuing corporate debt has become an integral part of the theory of corporate finance.

Structural bond pricing models value debt as a contingent claim on the firm's assets. This approach was pioneered by Merton(1974) and has since drawn considerable attention from practitioners and academics alike. An important feature of structural bond pricing models is that since all securities of a firm are treated as derivatives on the firm's assets, it is possible to use price information for one class of securities, typically equity, to infer the value of another, typically debt. Structural models normally call for parameters determining the behavior of the assets of the issuer, such as the asset volatility. One must also make assumptions concerning the capital structures of issuers. In addition, one must allow for correlation between assets and short rate term structure, when relevant. Empirical implementations of structural models have varied widely in their resolutions of these issues.

This paper focuses on the empirical analysis of credit risk based on a GARCH model structural approach. The remainder of this article is organized as follows. Section 2 is related literature review. Section 3 presents the basic GARCH risky debt valuation framework. Section 4 is valuing fixed-rate debt. Section 5 performs the analysis of the comparative static for credit spread. Section 6 explores the dynamic correlations among spreads and other macroeconomic variables. We construct a linear regression that expresses contemporaneous yield spreads as linear functions of the past histories of several variables (also see Duffie and Singleton (2003)). Section 7 summarizes the article and makes concluding remarks

## 2. Related Literatures Review

In the original models of Black and Scholes(1973) and Merton(1974), the asset-value process A is typically assumed to be risk-neutrally log normal distribution as follows:

$$\frac{dA_t}{A_t} = (r-\gamma)dt + \sigma dB_t^*, \tag{1}$$

where $\sigma$ is constant asset volatility, r is the constant instantaneous short rate and $\gamma$ is the constant cash payout rate.

Extending the one factor log-normal model in equation (1), the two factors structural model of Longstaff and Schwartz(1995) assume that

$$\frac{dA_t}{A_t} = (r_t-\gamma)dt + \sigma dB_t^*, \tag{2}$$

$$dr_t = \kappa(\mu - r_t)dt + \sigma_r dW_t^*, \tag{3}$$

where $W^*$ is a risk-neutral Brownian motion, with corr( $W_t^*$ , $B_t^*$ )= $\rho$ , i.e. the instantaneous correlation between $dB_t$ and $dW_t$ is $\rho dt$ . The risk neutral short rate model is thus of type introduced by Vasicek(1977). Kim et al.(1993) and Cathcart and El-Jahel(1998) considered a variant of this model in which the short rate process r is a one factor CIR process.

Empirical implementations of the one factor log-normal model have typically assumed that the firm is capitalized with common stock and one bond. The second that default occurs when $A_T$<D, where D is a default triggering boundary, usually estimated in terms of book liabilities. The third, default free short rate r is allowed to vary deterministically so as to capture the spot yield curve on the day valuation is undertaken. Finally, coupon bonds are priced as though they are a portfolio of zero coupon bonds corresponding to coupons and principal. When applying estimates of

two factors Longstaff and Schwartz (1995) model to corporate debt pricing, the empirical literature usually assumes that (1) default occurs at the first time that $A_t$ falls below a predetermined boundary, typically the face value of outstanding bonds, and (2) in the event of default, bondholders recover a constant fraction (1-$\omega$) of the face value. Again, risky coupon bonds are priced as a portfolio of risky zero coupon bonds.

More recently, Lyden and Sariniti(2000) implemented the Merton's model and found mean absolute errors in yield spreads of roughly 80 basis points. The model implied spreads were particularly low (bonds were overpriced) for small firms and long maturities. Lyden and Sariniti(2000) found that the Longstaff and Schwartz (1995) model performs roughly as well as the one factor model. Moreover, their findings for the Longstaff-Schwartz model were remarkably insensitive to correlation between A and r. Indeed, the Merton(1974) model dominated the two factors model when a common, aggregate recovery fraction w was used for all firms. Use of industry specific recovery values for individual issuers only worsened the fit. Eom, Helwege and Huang(2004) have undertaken the empirical analysis of structural models to date. They estimate the Black-Scholes-Merton and Longstaff-Schwartz models and find that all of these models have difficulty in accurately predicting credit spreads[1] and that the difficulties are not limited to underestimation. Their version of Black, Scholes and Merton model does underestimate spreads. However, the Longstaff and Schwartz model predicts spreads that are too large on average. More precisely, the model predicts excessive spreads for the riskiest bonds, and underestimates spreads on the safest bonds.

This paper seeks to propose a structural simple GARCH approach to valuing risky corporate debt that incorporates both default and interest rate risk. We attempt to derive simple analytical solution valuation expression for corporate debt. Trying to remedy the underestimation of credit spread based on Black, Scholes and Merton model and the exaggeration of credit spread prediction under the Longstaff and Schwartz(1995) model. The Black, Scholes and Merton model assumes the lognormal probability distribution of the asset value at any future time. Since volatility is the unobservable parameter in this model, the model gives the corporate debt as a function of volatility. According to the related literatures, the volatility is time varying. It is intuitively to allowing time dependence in asset value volatility. The constant volatility assumption is undoubtedly not reflecting the market reality. Several approaches to model the behavior of asset volatility variations have been proposed in

[1] Credit Spread is defined as the difference between the yields of a risky and a risk free bond with identical maturity dates and coupon rates.

the literature. The earlier one is the stochastic volatility model, see Hull and White(1987). The idea has been duplicated into the Longstaff and Schwartz(1995) model when pricing the corporate debt. Another volatility model appeals much popularity in recently is the generalized autoregressive conditional heteroskedasticity (GARCH) model, and many versions of the GARCH model are available. Duan(1995) proposed the version for the risk neutral behavior of the asset price that derivatives can be priced based on GARCH framework, also see Ritchken and Trevor(1999).

## 3. A preliminary GARCH model for risky bond valuation

A firm's asset portfolio, consisting of loans, traded securities and many other items, is refinanced by debt and equity. It is current practice today to judge soundless of a firm, by looking at accounting data, which are directly observable. The actual market value of the assets, that reveals more information on the firm's financial health, is not directly observable. This section describes the structural model, which is used to estimate market values of the firm's assets, which will then allow us to estimate the portfolio risk of a firm.

Consider a discrete-time economy and let $A_t$ designate the total value of the assets of the firm. Its one-period rate of return is assumed to be conditionally lognormally distributed under probability measure P. That is,

$$\ln \frac{A_t}{A_{t-1}} = r_t + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \sqrt{h_t}\varepsilon_t \tag{4}$$

where $\varepsilon_t$ has mean zero and conditional variance unity under measure P; and $\lambda$ can be interpreted as the unit risk premium. Under conditionally lognormality, one plus the conditionally expected rate of return equals $\exp(r_t + \lambda\sqrt{h_t})$. $r_t$ is the one-period risk-free rate of asset's value (continuously compounded). The dynamics of $r_t$ are given by

$$r_t - r_{t-1} = q(m - r_{t-1}) + \nu\xi_t \tag{5}$$

where q, m and $\upsilon$ are constants. This assumption as to the dynamics of r is drawn from the term structure model of Vasicek(1977). Although consistent with many of the observed properties of interest rates, these dynamics can allow negative interest rates. However, the probability of negative interest rates occurring is small for realistic parameter values. Vasicek(1977) provides the default-free bond price expression and it is given in the Appendix A for completeness. We further assume that $\varepsilon_t$ follows a GARCH(1,1) process of Bollerslev(1986) under measure P. Formally,

$\varepsilon_t \left| \phi_{t-1} \sim N(0, h_t) \right.$, under measure P, and

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1}(\varepsilon_{t-1} - c)^2 \tag{6}$$

where $\phi_t$ is the information set ($\sigma$-field) of all information up to and including time t; $\beta_0 > 0$, $\beta_1 \geq 0$, and $\beta_2 \geq 0$. To ensure covariance stationarity of the GARCH process, $\beta_1 + \beta_2$ is assumed to be less than 1. In words, the conditional variance is a linear function of past squared disturbances and the past conditional variance. Clearly, $h_t$ is predictable based on information set $I_t$.

In order to develop the GARCH risky debt pricing model, the conventional risk-neutral valuation relationship has to be generalized to accommodate heteroskedasticity of the asset value process. By the inference of Duan(1995), a pricing measure Q is said to satisfy that locally risk-neutral valuation relationship if measure Q is mutually absolutely continuous with respect to measure P, $\frac{A_t}{A_{t-1}} \left| I_{t-1} \right.$ distributes lognormally,

$E^Q(\frac{A_t}{A_{t-1}} \left| \phi_{t-1} \right.) = e^{r_t}$, and $Var^Q(\ln(\frac{A_t}{A_{t-1}}) \left| \phi_{t-1} \right.) = Var^P(\ln(\frac{A_t}{A_{t-1}}) \left| \phi_{t-1} \right.)$ almost surely with respect to measure P. In the above inference of locally risk-neutral valuation relationship, the conditional variances under two measures are required to be equal. This is useful due to one can observe and hence estimate the conditional variance under P. This property and the fact that the conditional mean can be replaced by the risk-free rate yield a good property model that does not locally depend on preference parameters. Duan(1995) proved under some combinations of preferences and distributions, the locally risk-neutral valuation relationship holds. Hence, under pricing measure Q,

$$\ln \frac{A_t}{A_{t-1}} = r_t - \frac{1}{2} h_t + \sqrt{h_t} \varepsilon_t^*, \tag{7}$$

where $\varepsilon_t^* = \varepsilon_t + \lambda$

$$h_t = \beta_0 + \beta_1 h_{t-1} + \beta_2 h_{t-1}(\varepsilon^*_{t-1} - c - \lambda)^2 \quad (8)$$

The conditional variance process under risk-neutralized pricing measure is not a GARCH process. The variance innovation is governed by one non-central chi-square random variables with one of degree of freedom, whereas the GARCH process under P can be seen as the process governed by one central chi-square innovations. Meanwhile,

$$r_t - r_{t-1} = q(m - r_{t-1}) + \nu\delta + \nu\xi^*_t \quad (9)$$

where $\xi^*_t = \xi + \delta$,

## 4. Valuing Fixed-Rate Debt

In this section, we try to infer valuation expressions for risky discount bonds and examine their implications for the term structure of credit spreads. Let $P(A_t,r_t,T)$ denote the price of a risky discount bond with maturity date T. The payoff on this contingent claim is 1 if default does not occur during the life of the bond, and (1-w) if it does. This payoff function can be expressed as

$$1 - wI_{\tau \le T} \quad (10)$$

where I is an indicator function that takes value one if A reach K during the life of the bond, and zero otherwise. More formally, I takes value one if the first-passage time $\tau$ of A to K is less than or equal to T.

The value of a risky discount bond $P(A_t,r_t,T)$, for maturity $\tau = T - t$, with constant recovery w, is given by

$$P(A_t,r_t,T) = w\, p(t,T-t) + p(t,T-t)(1-w)\, F(A/K,t, r_t, T) \quad (11)$$

where

$$F(t, r_t, T) = \sum_{i=1}^{n} f_i$$

$$f_1 = N(b_1),$$

$$f_i = N(b_i) - \sum_{j=1}^{i-1} f_j N(c_{ij}), i = 2,3,...,n,$$

$$b_i = \frac{-\ln(A/K) - U(iT/n,T)}{\sqrt{Q(iT/n)}},\ c_{ij} = \frac{U(jT/n,T) - U(iT/n,T)}{\sqrt{Q(iT/n) - Q(jT/n)}},$$

and

$$U(t,T) = (\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2})t + (\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3})\exp(-\beta T)(\exp(\beta t) - 1)$$

$$+(\frac{r}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3})(1 - \exp(-\beta t)) - (\frac{\eta^2}{2\beta^3})\exp(-\beta T)(1 - \exp(-\beta t)),$$

$$Q(t) = (\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2)t - (\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3})(1 - \exp(-\beta t)) + (\frac{\eta^2}{2\beta^3})(1 - \exp(-2\beta t))$$

and the default-free debt price is given by the Vasicek formula,

$$p(t,\tau) = \exp(G(\tau) - V(\tau)r(\tau)),$$

$$G(\tau) = (\frac{\eta^2}{2\kappa^2} - \frac{\theta}{\kappa})\tau + (\frac{\eta^2}{\kappa^3} - \frac{\theta}{\kappa^2})(\exp(-\kappa\tau) - 1) - (\frac{\eta^2}{4\kappa^3})(\exp(-2\kappa\tau) - 1)$$

$$V(\tau) = \frac{1 - \exp(-\kappa\tau)}{\kappa}$$

Proof is in the Appendix A.

The analytical form relates nothing more sophisticated that the standard normal distribution. We can observe that the function of a risky discount bond depends on asset value A and threshold value K for default merely by their ratio, namely A/K. From this point of view, one can think the ratio value play a role of proxy variable for the default rating of the corporation. An interesting feature of this is that the value of a risky debt can be measured without having to specify the values of A and K respectively. And one can simplify the practical operation of the function.

There are other intuitions on such a structure. From the equation (11), the first component in the right hand side shows the value of the bond would have if it were riskless. Another component represents a discount for the credit risk of the debt. It is worthy to recognize that the probability of a default F(A/K, t, $r_t$, T) under the risk-neutral measures may differ from the actual probability of a default.

Due to the ratio A/K can be seen as a sufficient statistic for credit risk, we do not need to condition on the pattern of the pattern of cash payments to be made prior to the maturity date of a debt in order to value the debt. Essentially, this is due to model assume that financial distress triggers the default of all of the corporation’s debt. An important institution of this is that coupon bonds can be valued as simple portfolios of discount bonds. The significant advantage is a main factor why this method is more attractive than the conventional method to pricing risky fixed-rate debt.

## 5. Comparative Static for Credit Spread

The duration of a risky discount bond need not be an increasing function of its maturity. For instance, for a middle level of credit risk, the duration of a discount bond can increase with time to maturity at first, level out, and then decrease with time to maturity. Indeed, for values of A/K and recovery rate w very close to unity, the influence of $r_t$ on the drift term can wipe the debt-price effect. Thus, a bond with risky can be an increasing function of $r_t$. As a result, the duration of very risky debt can actually get into negative.

The price of a risky bond is an increasing function of the state variable A/K. We can find the higher the value of state variable A/K, the further the corporation is from the default threshold and the lower the discount effect for default risk. After differentiating with respect to recovery rate, one can infer debt values are decreasing functions of w. One reason for this phenomenon is because a larger in w implies that the smaller price on a debt security in the event of default distress is larger.

Meanwhile, as time to maturity increases, the value of p(t,T-t) decrease and the potential probability of a default $F(A/K,t, r_t, T)$ increase. Both of these effects tend to decrease the value of the risky debt. Under such a logical dissection, risky debts are decreasing functions of time to maturity.

Generally, the value of a risky debt is a non-increasing function of market interest rate. Additionally, the sensitivity of the value to change in interest rate provides a evaluation of the duration of the bond. The duration of a risky debt is shorter than for a riskless treasury bond. We can evidence again by our model. The probable reason for this is that interest rate acts as two roles in the process of valuing risky debt. When interest rate is increasing will results in a lower value for default probability. Besides, an increase in interest rate will bring the drift term upward for asset value is higher. Hence, as interest rate increases, firm’s asset value is anticipated to shift away from the default threshold value at a faster rate, which decreases the credit risk.

Credit spread is defined as the difference between a riskless bond with identical maturity dates and coupon rates and the yields of a risky debt. Given the expression solution for risky fixed-rate bond, one can infer the credit spread. We get a picture that the term structure of credit spreads can be monotone increasing as well as hump shaped. Corresponding to the empirical study by Sarig and Warga (1989), which argues that the term structure of credit spreads is monotone increasing for debts with high credit ratings, and humped shaped for bonds with low tier of the credit ratings. Additionally, the differences of the credit spreads implied by this model are consistent with the general situations observed in bond markets.

To evaluate the influence of asset value and default-free interest rates on credit spreads, we consider the limiting expression for short maturity spreads. From the equation (11) could yield the instantaneous credit spread. In contrast to the extant literature, we can find that default-free interest rates have a direct effect on credit spread. For example, short-term credit spreads are insensitive to interest rates in Longstaff and Schwartz(1995). However, there is significant evidence that credit spreads are sensitive to changes in interest rates. Duffee (1998) suggests an inverse relation between interest rates and credit spreads. Our model is the one model consistent with the empirical observation on sensitivity of credit spreads to slides in the market interest rate. The direction of this relation is dependent on the duration gap between the corporations' interest-sensitive assets and liabilities. Moreover, one have to differentiate between short-term responses of credit spreads to interest rate changes as well as the long-term equilibrium adjustment. Specifically, we find that long-term interest rates also can impact positively short-term credit spreads.

## 6. The regression for credit spreads on macroeconomic variables

Credit spreads may show the phenomenon of substantial persistence over time. Thus, in developing a model of risk-neutral default intensities based on historical bond yield spreads, it is not sufficient to know the contemporaneous correlations among spreads, interest levels, and other variables that might influence default intensities. It is also helpful to have information about the temporal interactions among these variables. In order to explore the dynamic correlations among spreads and other macroeconomic variables, we estimate a linear equation that expresses contemporaneous yield spreads as a linear functions of the past histories of four variables: yield spreads (its own history, say, SPREAD), an index of consumer confidence (ConConf), the constant maturity treasury (CMT)[2] 30-year Treasury yield (CMT30), and the Standard & Poor's 500 stock index return (S&P500). Specifically,

$$SPREAD_t = \alpha_0 + \sum_{j=1}^{J} \beta_{1j} SPREAD_{t-j} + \beta_{2j} ConConf_{t-j} + \beta_{3j} CMT30_{t-j} + \beta_{4j} S\&P500_{t-j} + \varepsilon_t,$$

where $\varepsilon_t$ is the regression error. We then calculate the response over time in SPREAD to a "shock" in each of these four variables, holding the other three variables fixed. The size of each shock is equal to one standard deviation of the regression errors. For instance, in the case of response of SPREAD to its own shock, we compute an estimate of the standard deviation of $\varepsilon_t$ and then trace out the effects on SPREAD of a shock to $\varepsilon_t$ while setting the shocks to the other three variables equal to zero. In the above expression, we allow lagged values of all of the variables to affect SPREAD, and we suppose that the corresponding representations of ConConf, CMT30, and S&P500 have the same feature. Thus, if we change one of the variables today, it may affect all four variables over time through these dynamic interactions It is the timing and magnitude of these interactions that we are attempting to capture with our linear regression model. Collin-Dufresne et al.(2001) finds that a large proportion of variation in yield spreads is unexplained by the macro information included in their statistical analyses. Under GARCH risky debt pricing model, it seems very interesting to discuss and explore such phenomenon.

## 7. Conclusion

This paper developed a new GARCH framework for valuing risky corporate debt that incorporates both default risk and interest rate risk. We also derive an analytical form valuation expression for risky debt under this model setting. A significant property of our model is that it can be applied directly to value risky debt when there are many coupon payment dates or when the capital structure of the firm is relatively complex. Additionally, this approach allows us to relax the assumption of the absolute priority in debt claim, which underlies the traditional approach to valuing risky debt.

[2] At the end of each trading day, primary U.S. Treasury securities dealers report closing prices of the most actively traded bills, notes and bonds to the Federal Reserve Bank of New York. CMT indexes are computed from yields on these securities. The 30-year CMT yield is the average yield of the actively traded securities with a constant maturity of 30 years. The Federal Reserve publishes this index in its weekly statistics.

We also demonstrate that the term structure of credit spreads may have a variety of different shapes. Besides, we explore that the credit spreads are inversely movement to the level of interest rates. The main advantage of this model is that it is easily applied to all types of corporate debt securities and can be used to provide specific valuing and hedging results rather than just general implications.

Future research should focus on testing whether this model is able to explain the actual level of corporate bond yields using detailed cross-sectional and time-series data for individual bonds and one needs to model the effects of differences in the liquidity of traded bond on credit spreads..

Appendix:

**A. Vasicek (1977) solution for default-free bond:**

The price ofdefault-free bonds, $p(t,\tau)$, is

$$p(t,\tau)=\exp(G(\tau)-N(\tau)r(\tau)) \quad \text{(A1)}$$

where $\frac{\theta}{\kappa}$ is the long-term mean ate of interest, $\kappa$ is the speed at which the interest rate r approaches to its long-term mean, $\eta$ is the volatility of changes in the instantaneous default-free interest rate.

$$G(\tau)=(\frac{\eta^2}{2\kappa^2}-\frac{\theta}{\kappa})\tau+(\frac{\eta^2}{\kappa^3}-\frac{\theta}{\kappa^2})(\exp(-\kappa\tau)-1)-(\frac{\eta^2}{4\kappa^3})(\exp(-2\kappa\tau)-1)$$

$$N(\tau)=\frac{1-\exp(-\kappa\tau)}{\kappa}$$

and $\tau=T-t$

**B. Parameters estimation for the term structure model of Vasicek(1977):**

The Ornstein-Uhlenbeck model, used by Vasicek(1977) to specify the dynamics of the short term interest rate.The dynamic of the process is described by the stochastic differential equation

$$dr_t=\lambda(\mu-r_t)dt+\sigma dW_t \quad \text{(B1)}$$

$W_t$ is a standard Brownian motion.

This equation has a simple discrete time counterpart(See Gourierous,2001)

$$r_t=\mu[1-\exp(-\lambda)]+\exp(-\lambda)r_{t-1}+\sigma(\frac{1-\exp(-2\lambda)}{2\lambda})^{1/2}\varepsilon_t \quad \text{(B2)}$$

$\varepsilon_t$ is a standardized Gaussian white noise.

This equation corresponds to a AR(1) representation for the $r_t$ process.

We can easily apply the Maximum Likelihood method to the autoregressive model above.

Let us first reparametrize the AR(1) process as

$$r_t=\mu(1-\rho)+\rho r_{t-1}+\eta\varepsilon_t \quad \text{(B3)}$$

where $\rho=\exp(-\lambda)$

The MLE of the parameters $\mu, \rho, \eta$ are asymptotically independent and equivalent to

$$\hat{\mu}_T = \frac{1}{T}\sum_{t=1}^{T} r_t = \overline{r}_T$$

$$\hat{\rho}_T = \frac{1}{T}\sum_{t=1}^{T}(r_t - \overline{r}_T)(r_{t-1} - \overline{r}_T) / \frac{1}{T}\sum_{t=1}^{T}(r_t - \overline{r}_T)^2$$

$$\hat{\eta}_T^2 = \frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t^2$$

where the residuals are defined by $\hat{\varepsilon}_t = r_t - \overline{r}_T - \hat{\rho}_T(r_{t-1} - \overline{r}_T)$

Therefore, we can obtain

$$\hat{\lambda}_T = -\ln \hat{\rho}_T$$

$$\hat{\sigma}_T^2 = -\frac{2\ln \hat{\rho}_T}{1 - \hat{\rho}_T^2}\hat{\eta}_T^2$$

So we can infer all the parameters that we need.

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